Energy of Stable Half-Quantum Vortex in Equal-Spin-Pairing

Shokouh Haghdani[1,*] and Mohammad Ali Shahzamanian[1]

[1]*Department of Physics, Faculty of Sciences, University of Isfahan, 81744 Isfahan, Iran*

In the triplet equal-spin-pairing states of both ^3He-A phase and Sr_2RuO_4 superconductor, existence of Half-Quantum Vortices $(HQVs)$ are possible. The vortices carry half-integer multiples of magnetic quantum flux $\Phi_0 = hc/2e$. To obtain equilibrium condition for such systems, one has to take into account not only weak interaction energy but also effects of Landau Fermi liquid. Our method is based on the explanation of the HQV in terms of a $BCS-$like wave function with a spin-dependent boots. We have considered $\ell = 2$ order effects of the Landau Fermi liquid. We have shown that the effects of Landau Fermi liquid interaction with $\ell = 2$ are negligible. In stable HQV, an effective Zeeman field exists. In the thermodynamic stability state, the effective Zeeman field produces a non-zero spin polarization in addition to the polarization of external magnetic field.

PACS numbers:

1. INTRODUCTION

One view to the liquid phase of ^3He contains in normal and superfluid parts which at 3×10^{-3} Kelvin the superfluid part starts to be occurred mostly in triplet pairing[1,2]. The spin triplet pairing in the superconductor compound Sr_2RuO_4 is observed experimentally bellow 1.5 Kelvin[3].

The triplet pairing contains particles with the same spin directions that leads to spin current. The spin current leads to interesting and important phenomena like $HQVs$ in equal-spin-paring (ESP). Unlike common vortices, the half-quantum vortices contain half-integer multiplications of the flux quantum $\Phi_0 = hc/2e$. The origins of vortices in type-II superconductors and ^3He or ^4He superfluid are different. In the former case, the vortices are appeared in the presence of external magnetic field while appearance of the vortices in the latter cases is caused by the rotation of the vessel which ^3He or ^4He is contained. External magnetic field influences the vortices in ^3He too and can generate half-quantum vortices.

Vakaryuk and Leggett[4] have shown that in equal-spin-pairing state, the stability condition of half-quantum vortices is obtained when strong interactions also taken into account. The general method for calculating the strong interactions are presented by the Landau Fermi liquid theory. They have considered only $\ell = 1$ term in this interaction. In this work, $\ell = 2$ term is accounted and it is found that term with $\ell = 2$ is very small compared to $\ell = 0, 1$ terms.

2. THEORETICAL APPROACH

In the ESP state of a spin triplet condensate, the spin of particles in the Cooper pair is either aligned (up) or antialigned (down) with a common direction in the space which is called ESP axis[1,2]. Therefore Cooper pairs may be described via a linear superposition of states $|\uparrow\uparrow\rangle$ and $|\downarrow\downarrow\rangle$. The pairs are condensed in the same orbital states, we show them by $\varphi_\uparrow$ and $\varphi_\downarrow$. The many-body wave function which describes a system of $N/2$ pairs by $\varphi_\uparrow$ and $\varphi_\downarrow$ can be written as

$$\Psi_{ESP} = A\{[\varphi_\uparrow(\mathbf{r}_1, \mathbf{r}_2)|\uparrow\uparrow\rangle + \varphi_\downarrow(\mathbf{r}_1, \mathbf{r}_2)|\downarrow\downarrow\rangle]...[\varphi_\uparrow(\mathbf{r}_{N-1}, \mathbf{r}_N)|\uparrow\uparrow\rangle + \varphi_\downarrow(\mathbf{r}_{N-1}, \mathbf{r}_N)|\downarrow\downarrow\rangle]\}, \tag{1}$$

where A is the antisymmetrization operator with respect to particles coordinates $\mathbf{r}_i$ and spins. Here, for simplicity it will be assumed that the ESP state is neutral. Annular geometry with the radius R and the wall thickness d with $d/R \ll 1$ is considered, to prevent complications connected to the presence of the core of vortex. Therefore at zero temperature, the HQV state of the condensate can be described via[4]

$$\Psi_{HQV} = \exp\{\frac{i\ell_\uparrow}{2}\sum_{i=\uparrow}\theta_i + \frac{i\ell_\downarrow}{2}\sum_{i=\downarrow}\theta_i\}\Psi_{ESP}, \tag{2}$$

here θ_i is the azimuthal coordinate of the ith particle on the annulus and the spin axis is along the symmetry axis of the annulus. The integer ℓ_σ denotes the angular momentum of σ, the component of pair wave function.

The thermal stability of system is obtained via minimization of energy of the system by using the wave function in Eq. (2). In fact one needs full version of original Hamiltonian essentially. In the simplest case, one may consider BCS Hamiltonian, H_{BCS}, along with spin triplet pairing term; however the mentioned Hamiltonian H is unable to provide the thermodynamic stability of the HQV. In such systems, the stability condition is obtained when strong interactions taken into account. The general method for calculation the strong interactions are presented by the Landau Fermi liquid theory. Although the method was used for investigating the normal metals, the generalized version of the method is used for studying superconductor and superfluid[1]. Thus Hamiltonian of the system involves two parts; 1) BCS Hamiltonian and 2) Landau Fermi liquid effects ,H_{FL},

then $H = H_{BCS} + H_{FL}$. First, the expectation value of the weak coupling Hamiltonian with the state Eq. (2) is calculated. In this case, one can write it as a sum of three terms with different physical sources:

$$E_{BCS} = E_0 + E_S + T. \tag{3}$$

where E_0 is the energy contribution originated from the freedom internal degrees of Cooper pairs. This contribution is independent on the center of mass motion of the Cooper pairs i.e., on quantum numbers $\ell_\uparrow$ and $\ell_\downarrow$ and the magnetic field magnitude for the annulus radius R where is much larger than the BCS coherence length ξ_0[5]. In this work we assume a large enough annuls so that this term is ignorable.
The second term in the Eq. (3) is the energy of spin polarization of the system. We assume N_σ as the particles number with spin projection σ. one can define S as a projection of the total number spin polarization on the symmetry axis as $S \equiv (N_\uparrow - N_\downarrow)/2$. Therefor, spin polarization energy is obtained as

$$E_S = \frac{(g_S \mu_B S)^2}{2\chi_{ESP}} - g_S \mu_B \mathbf{B.S}, \tag{4}$$

where g_S is the gyromagnetic ratio for particles and χ_{ESP} is the ESP state spin susceptibility. It is important that the spin polarization S, is a variational parameter and the actual value of S is obtained by minimization of the energy.
The third term in the Eq. (3) is the kinetic energy of the currents circulating in the system. We introduce the new parameters as[4]:

$$\ell_{s\Phi} \equiv \frac{\ell_\uparrow + \ell_\downarrow}{2} - \frac{\Phi}{\Phi_0}, \ell_{sp} \equiv \frac{\ell_\uparrow - \ell_\downarrow}{2}, \tag{5}$$

where Φ is the total flux through the annulus. By using the above introduced parameters, T takes the following form:

$$T = \frac{\hbar^2}{8m^* R^2}\{(\ell_{s\Phi}^2 + \ell_{sp}^2)N + 4\ell_{sp}\ell_{s\Phi}\}, \tag{6}$$

where m^* is the effective mass of particles containing the Fermi liquid corrections. It is related to the bare mass of particles m by the usual relation of Fermi liquid theory as $m^* = m(1 + F_1/3)$[1,2]. In Eq. (6) the first term in the brackets is constant. Because it is proportional to the total number of particles $N \equiv N_\uparrow + N_\downarrow$ and given values of $\ell_{s\Phi}$ and ℓ_{sp}. The second term generates an effective Zeeman field in the HQV state due to proportional to the spin polarization S. The magnitude of this field and hence the value of the thermal equilibrium spin polarization should be obtain by energy minimization[4].
For making an HQV stable, the strong coupling effects should be considered[1,6]. Also it needs to be accounted the change of Fermi liquid energy E_{FL} due to the presence of spin and momentum currents in the HQV state. These currents are created by the spin dependent boost of Eq. (2). We use the standard formalism of Fermi liquid theory and investigate $\ell = 2$ order effects of Landau Fermi liquid. By considering the mentioned assumptions, we finally reach at the lengthly expression as follow:

$$\begin{aligned} E_{FL} = \quad & \frac{1}{2}(\frac{dn}{d\varepsilon})^{-1} Z_0 S^2 + \\ & \frac{N^{-1}\hbar^2}{8m^* R^2}\frac{1}{3}\{(\ell_{s\Phi}^2 F_1 + \ell_{sp}^2 \frac{Z_1}{4})N^2 + 4(\ell_{sp}^2 F_1 + \ell_{s\Phi}^2 \frac{Z_1}{4})S^2 + 4\ell_{s\Phi}\ell_{sp}(F_1 + \frac{Z_1}{4})SN\} \\ & + \frac{N^{-1}\hbar^4}{64 m^* p_F^2 R^4}\{[(F_2 + \frac{Z_2}{4})(\ell_{s\Phi}^2 + \ell_{sp}^2)^2 + \frac{Z_2}{2}\ell_{s\Phi}^2\ell_{sp}^2]N^2 + [(16F_2 + 2Z_2)\ell_{s\Phi}^2\ell_{sp}^2] \times \\ & S^2 + [(8F_2 + 2Z_2)\ell_{s\Phi}\ell_{sp}(\ell_{s\Phi}^2 + \ell_{sp}^2)]SN\} - \frac{N^{-1}p_F^2}{12m^*}(F_2 N^2 + Z_2 S^2), \end{aligned} \tag{7}$$

where $(dn/d\varepsilon)$ is the density of states at the Fermi surface and Z_0, Z_1, Z_2, F_0 and F_1 are Landau parameters. The first term in the above equation, proportional to Z_0, is the energy cost generated by a spin polarization and the rest explains Fermi liquid corrections caused by the presence of currents.
Now we obtain the equilibrium spin polarization in the HQV state by minimizing the total energy $E_{BCS} + E_{FL}$ with respect to S. Then S can be written as

$$S = (g_s \mu_B)^{-1} \chi B^{'}, \tag{8}$$

where χ is the spin susceptibility of the system. For $^3He - A$ the value of χ is approximately 0.37 times smaller than the normal state susceptibility at low temperatures[6]. We also find χ as:

$$\begin{aligned} \chi^{-1} = \frac{1}{\chi_{ESP}} + (\frac{dn}{d\varepsilon})^{-1} Z_0 (g_S\mu_B)^{-2} + \frac{N^{-1}\hbar^2}{m^* R^2 (g_S \mu_B)^2}(\ell_{sp}^2 \frac{F_1}{3} + \ell_{s\Phi}^2 \frac{Z_1}{12}) \\ + \frac{N^{-1}\hbar^4}{16 m^* p_F^2 R^4 (g_S\mu_B)^2}\ell_{sp}^2\ell_{s\Phi}^2(8F_2 + Z_2) - \frac{N^{-1}p_F^2 Z_2}{6m^*(g_S\mu_B)^2}. \end{aligned} \tag{9}$$

The Zeeman field $B^{'}$ is involved two parts: 1) the external Zeeman field B and 2) the effective Zeeman field B_{eff}:

$$B^{'} = B + B_{eff}, \tag{10}$$

The effective Zeeman field B_{eff} is due to the present of spin currents:

$$B_{eff} = -\frac{\hbar^2 (g_S \mu_B)^{-1}}{2m^* R^2} \ell_{sp} \ell_{s\Phi} \{1 + \frac{F_1}{3} + \frac{Z_1}{12} + \frac{\hbar^2}{16 p_F^2 R^2}(4F_2 + Z_2)(\ell_{s\Phi}^2 + \ell_{sp}^2)\}. \tag{11}$$

The effective Zeeman field is a periodic function of the total external magnetic flux Φ with a period equal to Φ_0. The sign of the effective field is altered when the total magnetic field is equal to half-integer values of the flux quantum[4].
Finally, the energy of the system $E \equiv \langle H \rangle$ can be obtained by inserting the value of S, according to Eq. (8) - Eq. (11), to the relation of the total energy and ignoring the internal energy contribution. Then one can write:

$$E = -\frac{1}{2}\chi B^{'} + \frac{\hbar^2 N}{8mR^2}\{\ell_{s\Phi}^2 + \ell_{sp}^2 \frac{1 + \frac{Z_1}{12}}{1 + \frac{F_1}{3}}\} + \frac{\hbar^4 N}{64 m p_F^2 R^4}\{(\ell_{s\Phi}^2 + \ell_{sp}^2)^2 \frac{F_2 + \frac{Z_2}{4}}{1 + \frac{F_1}{3}} + \ell_{s\Phi}^2 \ell_{sp}^2 \frac{Z_2}{1 + \frac{F_1}{3}}\}. \tag{12}$$

The contribution of the spin polarization to the energy (first term of Eq. (12)) is small for reasonable values of the external magnetic field. The third term of the equation is related to $\ell = 2$ which compare to the second term is order of $\hbar^2/2mR^2\varepsilon_F$. Evidently, one can ignore them for analyzing the stability of HQVs. The stability region of the HQVs depends the $(1 + Z_1/12)/(1 + F_1/3)$ that is, the ratio of superfluid spin density to superfluid density ρ_{sp}/ρ_s[1]. The criteria of stability is directly obtained by minimizing of Eq. (12) and leads to $\rho_{sp}/\rho_s \prec 1$[7]. In the $^3He - A$, the ratio is less than unity for all temperatures bellow critical temperature and then HQVs are possible for a large limit of the phases diagram[7].

3. CONCLUSIONS

In the equal-spin pairing condensation of HQV an effective Zeeman field B_{eff} exists. In the thermodynamic stability state, the effective Zeeman field produces a non zero spin polarization in addition to the polarization of external magnetic field B. The thermal stability of system is obtained via minimization of energy of the system by using the wave function and the total Hamiltonian. The best Hamiltonian of the system involves two parts; 1) BCS Hamiltonian and 2) Landau Fermi liquid Hamiltonian. The effects of $\ell = 2$ terms of Landau Fermi liquid have been considered in this paper. The third term in Eq. (12) contains Landau parameters Z_2 and F_2, which compare to the second term is order of $\hbar^2/2mR^2\varepsilon_F$. This quantity with $R = 0.1$ micron takes approximately 10^{-7}. Therefore, one can omit this term in Eq. (12). Consequently for the stability condition of HQVs, it is sufficient that the second term of Eq. (12) be considered and $\frac{\rho_{sp}}{\rho_s} < 1$ is obtained for stability condition.

* Electronic address: shokouhhaghdani@gmail.com

1 Leggett A J 1975 *Rev. Mod. Phys.* **47** 331-414.
2 Vollhardt D and Wolfle P 1990 *The Superfluid Phases of Helium 3* (London: Taylor and Francis).
3 Mackenzie A P and Maeno Y 2003 *Rev. Mod. Phys.* **75** 657-712.
4 Vakaryuk V and Leggett A J 2009 *Phys. Rev. Lett.* **103** 057003.
5 Vakaryuk V 2008 *Phys. Rev. Lett.* **101** 167002.
6 Leggett A J 1965 *Phys. Rev. Lett.* **14** 536-537.
7 Volovik G E 1999 *JETP Lett.* **70** 792.